Indian Statistical Institute, Bangalore

B. Math.

First Year, Second Semester

Algebra II (Linear Algebra)

Mid-term Examination — Date : February 22, 2010

Maximum marks: 100 — Time: 3 hours

1. Let T be the linear map from $I\!R^3$ to $I\!R^2$ defined by

$$T\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2y_1 + y_2 \\ 3y_3 - 2y_1 \end{pmatrix}.$$

Let $\mathcal{B}_1, \mathcal{C}_1$ be standard bases of $I\!R^3, I\!R^2$ respectively. Let $\mathcal{B}_2 = \{\alpha_1, \alpha_2, \alpha_3\}$, where

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and let $\mathcal{C}_2 = \{\beta_1, \beta_2\}$, where

$$\beta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \beta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Compute the matrices ${}_{\mathcal{C}_1}[T]_{\mathcal{B}_1}$, ${}_{\mathcal{C}_1}[T]_{\mathcal{B}_2}$, ${}_{\mathcal{C}_2}[T]_{\mathcal{B}_1}$, and ${}_{\mathcal{C}_2}[T]_{\mathcal{B}_2}$. [20]

2. Show that row rank of a matrix is same as its column rank (on any field). [20]

3. Let $\mathcal{W}$ be the vector space of all 2×2 real matrices on $I\!R$ with respect to usual operations. Let $\mathcal{M} = \{A \in \mathcal{W} : \ \text{trace} \ (A) = 0\}$. Show that $\mathcal{M}$ is a subspace of $\mathcal{W}$. Compute the dimension of $\mathcal{M}$ and obtain a basis for $\mathcal{M}$. [15]

4. Let $\mathcal{L}$ be the vector space of real valued continuous functions on $[-1, 1]$. Take

$$\mathcal{J} = \{f \in \mathcal{L} : f(-t) = f(t) \quad \forall t \in [-1, 1]\};$$

$$\mathcal{K} = \{f \in \mathcal{L} : f(-t) = -f(t) \quad \forall t \in [-1, 1]\}.$$

Show that $\mathcal{J}, \mathcal{K}$ are subspaces of $\mathcal{L}$ and $\mathcal{L}$ is the vector space direct sum of $\mathcal{J}$ and $\mathcal{K}$.

[15]

5. Solve the following set of linear equations by transforming the associated matrix to reduced echelon form:

$$\begin{aligned} 2x_1 &= 6 \\ 2x_1 + x_2 + x_3 + x_4 &= 7 \\ 4x_1 - 2x_3 &= 12 \\ -x_1 + 3x_2 &= 0. \end{aligned}$$

[20]

[P.T.O.]

6. Let $\mathcal{H}$ be a finite dimensional inner product space over $\mathbb{C}$. Show that $S : \mathcal{H} \to \mathbb{C}$ is a linear map if and only if there exists $z \in \mathcal{H}$ such that

$$S(x) = \langle z, x \rangle,$$

for all $x \in \mathcal{H}$. (Hint: You may fix an ortho-normal basis for $\mathcal{H}$). [10]

7. (Bonus question) Let $\mathcal{V}$ be a finite dimensional inner product space over $\mathbb{C}$ with dimension $(\mathcal{V}) = n$. Let $T : \mathcal{V} \to \mathcal{V}$ be a linear map and $x \in \mathcal{V}$ is a vector with $\|x\| = 1$. Suppose T is self-adjoint and $\{x, Tx, T^2x, \ldots, T^{n-1}x\}$ is a linearly independent set. Show that there exists an ordered ortho-normal basis $\mathcal{B} = \{z_1, z_2, \ldots, z_n\}$ of $\mathcal{V}$ with $z_1 = x$, such that $A :=_{\mathcal{B}}[T]_{\mathcal{B}}$ is *tri-diagonal*, that is, $A_{ij} = 0$ for $|i - j| > 1$. (Hint: Use Gram-Schmidt orthogonalization)

[10]